Simply Supported Non Prismatic Beam Resting On Variable Elastic Foundation Carrying Travelling Harmonic Loads

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Abstract
The dynamic response under travelling loads of simply supported non prismatic beam resting on variable elastic foundation is investigated in this paper. The movement of people and goods has always been an important and fundamental component of human endeavours since creation; and development of sophisticated machines and vehicles due to technological advancement has continued to be more complex as the population increases. The safety of lives and properties cannot be overemphasized which makes the study of analysis of moving loads on structural members draws the attentions of researchers in Mechanical, Civil, Aerospace Engineering, and Mathematical Physics and so on. Common examples of such structures include; beam, plates and shells while moving loads include moving trains, trucks, cars, bicycles cranes and so on. Governing equations are solved using Generalized Galerkin’s technique to reduce the fourth order partial differential equations governing the motion of the beam with variable and singular coefficients to a sequence of second-order non-homogeneous ordinary differential equations. Method of Laplace Integral transforms is employed to solve this initial valued problem to obtain the desired approximate solutions of the reduced equations for the transverse displacement response of the beam dynamical problem. The analytical procedure illustrated practical engineering interest in which the effects of important parameters, axial force and foundation stiffness are investigated in depth. Resonance phenomenon of the vibrating system is carefully investigated and the condition under which this may occur is clearly scrutinized. This will ensure stability and durability of structures carrying loads and guarantee safety of lives and properties. The results presented in this paper will form basis for a further research work in this field. In this study, only simply supported boundary condition is considered. The case of beam under accelerating loads is left for further investigations. The numerical results are presented in plotted curves. Analyses show that higher values of the axial force and foundation stiffness decrease the transverse displacement response of the non prismatic beam under the action of travelling loads resting on variable elastic foundation.

Keywords: variable foundation, non-prismatic beam, harmonic loads, vibrating system differential equation

INTRODUCTION
Over the decades, problems involving vibration in many areas such as mechanical, civil and aerospace engineering; wave loading of offshore platforms, cabin noise in aircrafts, earthquake and wind loading of cable stayed bridges and high rise buildings, performance of machine tools – to pick only few random examples has attracted the attention of several researchers in Engineering, Applied Physics and Applied Mathematics. Notable among such researchers are Frybal (1972), Oni and Adedowole (2008) and so on.

Extensive work has been done on this class of dynamical problems when the structural members have uniform cross-sections. The vibration of non-uniform beams is also a subject of considerable scientific and practical interest that has been studied extensively, and is still receiving attention in literature. Worthy of mention in this area of work is the work of Kolousek (1967).

Among the recent works is also the work of Ojih et al (2013) investigated the dynamic response of non uniform Rayleigh beam resting on Pasternak foundation and subjected to concentrated loads travelling at constant velocity with simply supported boundary condition. Taha and Abohadima (2008) investigated Mathematical model for vibrations of non-uniform flexural beams.

Very recently, Adedowole (2016) worked on flexural motions under moving distributed masses of Beam-type structures on Vlasor foundation and having time dependent boundary conditions. The author (2017) also consider flexural vibration of non prismatic Rayleigh beam with non uniform prestress under concentrated loads moving with variable velocity. Omolofe and Ogunyebi (2016) studied the dynamic behaviour of a rotating Timoshenko beam when under the actions of a variable magnitude load moving at non-uniform speed.
This present study therefore concerns the dynamic characteristics of simply supported non prismatic beam resting on variable elastic foundation carrying travelling harmonic loads. It is assumed that the speed at which the travelling load traverses the structural elements is constant.

**Definition of the Problem**

The equation governing the transverse displacement \( W(x,t) \) of non prismatic beam when it is resting on a variable elastic foundation and traversed by travelling harmonic loads is the fourth order partial differential equation given by Frybal (1972).

\[
\mu(x) y_{x}(x,t) + C(x) y_{xx}(x,t) + F(x) y(x,t) = P(x,t) + D_{\alpha}(x,t) + N_{n} y_{x}(x,t) \tag{1}
\]

Where \( P(x,t) \) is the moving concentrated forces acting on the beam, \( \mu \) is the mass of the beam per unit length \( L, C \) is the material damping intensity, \( y(x,t) \) is the vertical response of the beam, and \( t \) is time. The flexural moment acting on the beam across section is related to the vertical response as

\[
D_{\alpha}(x,t) = -EI \left( z_{\alpha}(x,t) \right) \tag{2}
\]

Where \( EI \) is the flexural rigidity of the beam, \( E \) is is the young modulus

The distribution of the non uniform characteristics may be assumed as power functions Taha (2008). The parameters \( \Psi \) and \( n \) are used to approximate the actual non uniformity of the beam given as

\[
I(x) = I_{\alpha}(1 + \Psi x)^{n+2}, \quad \mu(x) = \mu_{\alpha}(1 + \Psi x)^{n} \tag{3}
\]

Where \( I(x) \) is the flexural moment of inertia of the beam, \( I_{\alpha}, \mu_{\alpha} \) and \( c_{\alpha} \) are the beam characteristics at \( x = 0 \).

**The Boundary Conditions**

For a simply supported beam whose length is \( L \), the vertical displacement at the beam the ends are given as:

\[
y(0,t) = y(L,t) = 0, \quad y_{\alpha}(0,t) = y_{\alpha}(L,t) = 0 \tag{4}
\]

It is assumed that the initial conditions are

\[
y(x,0) = 0 = y_{\alpha}(x,0) \tag{5}
\]

**CASE 1: Analysis of non prismatic beam under constant magnitude**

In this paper, we adopt the example in Omolofe (2016) and define the variable elastic foundation \( F(x) \) as

\[
F(x) = F_{s}(4x-3x^{2} + x^{3}) \tag{6}
\]

where \( F_{s} \) is the elastic foundation constant

Furthermore, the constant vertical excitation acting on the beam is chosen as

\[
P(x,t) = P\delta(x-v_{i}t) \tag{7}
\]

\[\delta(x-v_{i}t) \] is the Dirac function \( \delta(x) \) which represents concentrated force acting at point \( x = 0 \) is defined as:

\[
\int_{-\infty}^{x} \delta(x) dx = 1, \int_{-\infty}^{x} \delta(x-a)f(x) dx = f(a) \tag{8}
\]

where \( v_{i} \) is the velocity of the \( i^{th} \) particle of the system, \( t \) is the travelling time substituting equations (2), (3), (4), (5), (6) & (7) into equation (1)

\[
k_{1} \left[ \frac{\partial^{2}}{\partial x^{2}} \left[ (1 + \Psi x)^{3} y_{xx}(x,t) \right] \right] + (1 + \Psi x) y_{n}(x,t)
-k_{2} y_{xx}(x,t) + k_{1} (1 + \Psi x) y_{x}(x,t) + k_{4} (4x - 3x^{2} + x^{3}) y(x,t)
= k_{3} \delta(x-v_{i}t) \tag{9}
\]

A closed form solution to the fourth order Partial Differential Equation (1) governing the motion of the non prismatic beam under the actions of moving force, does not exist. It is desirable to obtain some vital information about the vibrating system.

**SOLUTION TECHNIQUE AND PROCEDURE**

To solve the beam problem above in equation (9), we shall use an elegant solution technique called Galerkin’s method. The equation of the motion of an element of the beam is generally symbolically written in the form.

\[
\Phi y(x,t) - P(x,t) = 0 \tag{10}
\]

where \( \Phi \) is the differential operator with variable coefficients, \( y(x,t) \) is the beam displacement, \( P(x,t) \) is the load acting on the beam, \( x \) and \( t \) are spatial coordinates and time respectively. The solutions of the system of equation (9) is expressed as

\[
y(x,t) = \sum_{i=1}^{n} y_{i}(t) Q_{i}(x) \tag{11}
\]

where \( y_{i}(t) \) are coordinates in modal space and \( Q_{i}(x) \) are the normal modes of free vibration written as

\[
Q_{i}(x) = \sin \theta_{x} x + A_{i} \cos \theta_{x} x + B_{i} \cosh \theta_{x} x + C_{i} \sinh \theta_{x} x \tag{12}
\]

where \( A_{i}, B_{i} \) and \( C_{i} \) are the space and amplitude of the vibration. For a simply supported beam, we have
where

Thus, for a beam with simple supports at both ends, equation (11) takes the form

\[ Q_i(x) = \sin \frac{i\pi x}{L} \]

Thus in view of equation (12) the transverse displacement response of a simply supported elastic beam, using an assumed mode method can be written as

\[ y(x,t) = \sum_{i=1}^{n} z_i(t) \sin \frac{i\pi x}{L} \]

Substituting (13) into the governing equation (9) and after some simplifications and arrangements one obtains

\[
k_1 \left[ \left( 1+\Psi_1 x + 3\Psi_2 x^2 + 3\Psi_3 x^3 \right) \frac{d^4}{dx^4} \sum_{i=1}^{n} z_i(t) \sin \frac{i\pi x}{L} \right] + \left( 6\Psi_2 x + 6\Psi_3 x^2 \right) \frac{d^3}{dx^3} \sum_{i=1}^{n} z_i(t) \sin \frac{i\pi x}{L} \\
+ \left( 6\Psi_3 x \right) \frac{d^2}{dx^2} \sum_{i=1}^{n} z_i(t) \sin \frac{i\pi x}{L} \\
- k_2 \frac{d^2}{dx^2} \sum_{i=1}^{n} z_i(t) \sin \frac{i\pi x}{L} \\
\[ = k_3 \delta(x-v_it) \]
\]

where

\[ z_i \] on the left hand sides of equation (14) are required to be orthogonal to the function \( \sin \frac{j\pi x}{L} \). Thus,

\[
\left[ \sum_{i=1}^{n} k_1 \left[ h_1(x) \left( \frac{i\pi}{L} \right)^4 \cos \frac{i\pi x}{L} - h_1(x) \left( \frac{i\pi}{L} \right)^3 \right] \right] z_i(t) \\
- h_3(x) \left[ \frac{(i\pi)^2}{L} \right] \sin \frac{i\pi x}{L} z_i(t) \\
+ h_4(x) \left[ k_2 z_i(t) \sin \frac{i\pi x}{L} + \ddot{z}_i(t) \sin \frac{i\pi x}{L} \right] \\
+ k_4 \left( 4x - 3x^2 + \dot{x} \right) \dddot{z}(t) \sin \frac{i\pi x}{L} \\
\]

Further rearrangements and simplifications of equation (15) we obtain

\[ b_1(i, j)\dot{y}_1(t) + b_2(i, j)\dot{y}_i(t) + b_3(i, j)\dot{y}_i(t) = k_5 \sin \frac{j\pi t}{L} \]

where

\[ b_1(i, j) = k_i [a_i - a_j - a_j] + a_i + a_j \]

\[ b_2(i, j) = [L_1 + \Psi I_2, b_3(i, j) = k_3 \frac{L}{2} \]

Equation (16) is the second order ordinary differential equation with constant coefficient to a transformation

In what follow we subject the system of ordinary differential equation (16) to a Laplace transform defined as

\[ \mathcal{L} \left[ f(t) \right] = \int_{0}^{\infty} e^{-st} f(t) dt \]

In conjunction with the initial conditions defined in (5), yields the following algebraic equation

\[ \left[ b_1(i, j)s^2 + b_2(i, j)s + b_3(i, j) \right] y_i(s) = k_5 \frac{s}{s^2 + \theta^2} \]

Subjecting equation (20) for further simplification yields

\[ y_i(S) = \frac{k_5}{(\alpha - \beta)} \left( \frac{\theta}{S^2 + \theta^2} \right) \left( \frac{1}{S - \alpha} \right) - \frac{\theta}{S^2 + \theta^2} \]

Where

\[ \theta = \left( \frac{j\pi t}{L} \right), \quad \alpha = -b_2 + \sqrt{b_2^2 - 4b_1b_4} \]

\[ \beta = -b_2 - \sqrt{b_2^2 - 4b_1b_4} \]

So that the Laplace inversion of each term of the RHS (21) is the convolution of \( f_i \)’s and \( g \) defined by

as \( f_i * g = \int_{0}^{\infty} f_i(t-u) g(u) du \), \( i=1,2 \)

Thus the Laplace inversion of (21) is given by

\[ y_i(t) = \mathcal{L}^{-1} \left[ e^{\frac{\beta t}{d_2} - \frac{\theta t}{d_1}} \right] \int_{0}^{\infty} \left[ e^{\frac{\beta t}{d_2} - \frac{\theta t}{d_1}} y_i(S) \right] dS \]

Further simplification of equation (24) which on inversion yields
\begin{equation}
y(x,t) = \sum_{n=1}^{\infty} \frac{e^{\omega n}}{\alpha(\omega^2 + \alpha^2)} \left( \theta e^{-\alpha \omega} - \alpha n \cos \omega t - \alpha \sin \omega t \right)
\end{equation}

\begin{equation}
\left( \theta - \frac{e^{\beta \omega}}{\beta(\omega^2 + \beta^2)} \right)
\left( \theta - e^{-\beta \omega} \{ \theta \cos \omega t - \beta \sin \omega t \} \right) = \sin \frac{m \pi x}{L}
\end{equation}

Equation (25) represents the displacement response of the non-prismatic Bernoulli beam resting on variable elastic foundation under the action of fast moving concentrated forces.

**CASE II Dynamics behaviour of a non prismatic beam subjected to Harmonic variable magnitude loads.**

In this section, the load \( P(x,t) \) is given as

\begin{equation}
P(x,t) = P \cos \Omega \delta(x-v t)
\end{equation}

where \( \Omega \) is the circular frequency of the harmonic force and all parameters are as defined previously. Substituting equation (26) in equation (1), vibration of the beam is then described by the equation

\begin{equation}
\mu(x) \ddot{y}(x,t) + C(x) y(t) + F(x) y(x,t)
\end{equation}

\begin{equation}
= P \cos \Omega \sin \theta
\end{equation}

Equation (27) is the governing equation describing the motion of non-prismatic beams subjected to fast moving loads of varying magnitude. Following the same arguments as in the previous section without loss of generality, considering only the \( m \)th particle of the dynamical system (27) yields

\begin{equation}
b_{1}(m,j) \ddot{y}_{m}(t) + b_{2}(m,j) \dot{y}_{m}(t) + b_{3}(m,j) y_{m}(t)
\end{equation}

\begin{equation}
= P \cos \Omega \sin \theta
\end{equation}

Equation (28) is analogous to equation (16), thus subjecting equation (28) to Laplace transform in conjunction with the boundary conditions stated (5) and using convolution theory we obtain

\begin{equation}
z_{m}(t) = P \left[ \frac{e^{\omega n}}{\alpha(\gamma_{1} + \alpha^2)} \left( \gamma_{1} - e^{-\alpha \omega} \{ \alpha \sin \gamma_{1} t + \gamma_{1} \cos \gamma_{1} t \} \right)
\end{equation}

\begin{equation}
- \frac{e^{\omega n}}{\alpha(\gamma_{2} + \alpha^2)} \left( \gamma_{2} - e^{-\alpha \omega} \{ \alpha \sin \gamma_{2} t + \gamma_{2} \cos \gamma_{2} t \} \right)
\end{equation}

\begin{equation}
- \frac{e^{\beta \omega}}{\beta(\gamma_{1}^2 + \beta^2)} \left( \gamma_{1} - e^{-\beta \omega} \{ \beta \sin \gamma_{1} t + \gamma_{1} \cos \gamma_{1} t \} \right)
\end{equation}

\begin{equation}
- \frac{e^{\beta \omega}}{\beta(\gamma_{2}^2 + \beta^2)} \left( \gamma_{2} - e^{-\beta \omega} \{ \beta \sin \gamma_{2} t + \gamma_{2} \cos \gamma_{2} t \} \right)
\end{equation}

Which on inversion yields

\begin{equation}
y_{m}(x,t) = \sum_{m=1}^{\infty} z_{m}(t) \sin \frac{m \pi x}{L}
\end{equation}

Equation (30) represents the transverse displacement response of non-prismatic beam resting on variable elastic foundation under actions of harmonic variable magnitude moving concentrated loads.

**DISCUSSION ON THE CLOSED FORM SOLUTION**

When elastic beam resting on variable foundation such as this is studied, it desirable to examine the phenomenon of resonance. For the resonance conditions of simply supported boundary conditions considered, equation (25) clearly shows that the non-prismatic elastic beam resting on variable elastic foundation and traverse by moving concentrated loads with uniform speed reaches a state of resonance whenever

\begin{equation}
\alpha = \beta, \quad \alpha^2 = -\theta^2 \quad \text{or} \quad \beta^2 = -\theta^2
\end{equation}

While equation (29) shows that the same beam under the action of harmonic variable magnitude moving loads will experience resonance effects whenever

\begin{equation}
\alpha = \beta, \quad \alpha^2 = -\gamma_{1}^2 \quad \text{or} \quad \beta^2 = -\gamma_{2}^2
\end{equation}

**NUMERICAL RESULTS AND DISCUSSION**

We shall illustrate the analysis proposed in this paper by considering a homogenous beam of modulus of elasticity \( E = 3.1 \times 10^{11} \text{ N/m}^2 \), the moment of inertia \( I = 2.87698 \times 10^{3} \text{ m}^4 \), the beam span \( L=12.192 \text{ m} \), \( \Psi = 0.025 \) and the mass per unit length of the beam \( \mu = 2758.291 \text{ Kg/m} \). The loads travel at velocity of 8.123 m/s. Furthermore, the values of \( F_s \) is varied between 5000N/m and 10000N/m, the values of \( N_v \) varied between 0N and 2.0 \times 10^7 N.

![Figure 1: Response of a beam subjected to constant magnitude loads for various values of \( F_s \)](image1)

![Figure 3: Deflection profile of a beam under Harmonic variable magnitude loads for various values of \( F_s \)](image3)
Figure 1 depicts the deflection profile of a simply supported non prismatic beam under the action of constant magnitude moving loads at constant velocity for various values of foundation stiffness and for fixed value of axial force $N_o=4000N$. The figure shows that as foundation stiffness $F_s$ increases, the deflection profile of the non prismatic beam decreases. Similar results are obtained when the beam is subjected to variable harmonic magnitude as shown in figure 3.

Figure 2 displays the deflection profile of the simply supported beam under the action of traveling concentrated forces when traveling loads are of constant magnitudes. For various values of axial force $N_o$. The figure shows that as $N_o$ increases the deflection of the non prismatic beam decreases. The same results are obtained when the simply supported beam is traversed by traveling load are of variable harmonic magnitude in figure 4.

Finally, figure 5 presents the comparison of the displacement response of moving constant and harmonic variable moving loads for fixed values foundation stiffness $F_s=4000$ N/m$^3$ and axial force $N=4000$ N.

CONCLUSION

The problem of the dynamic behaviour under moving concentrated loads of non prismatic beam on variable elastic foundation is considered in this work. The governing equation is a fourth order partial non-homogenous differential equation. The objective of the work has been to study the problem of the dynamic response to moving concentrated of the beam on variable foundation stiffness. In solving the governing fourth order partial differential equation, we employed a technique based on the Generalized Galerkin method and convolution theory. These solutions are analyzed and resonance conditions are obtained for the problem. Numerical analysis for both moving constant and harmonic variable moving loads problems are carried out and the results are presented in plotted curves.

REFERENCES


