On Maximization of Strehl-Ratio and Minimization of Second-Order Moment in the Green’s Functions of Apodised Optical Systems

A Srisailam\(^1\) and C Vijender\(^2\)

\(^1\)Department of Mathematics, Osmania University, Hyderabad-500007, A.P., India.  
\(^2\)Department of Mathematics, Srinidhi Engineering College, Hyderabad, A.P., India

**Corresponding Author:** A Srisailam

Strehl-Ratio is an important Point-Image quality-assessment parameter especially when the optical system has some aberrations. The Second-Order Moment determines the central peak intensity. In this paper, we evaluated these parameters of an optical system with apodised Bartlett Window Functions. Our studies show that for lower values of the apodisation parameter these filters give well results.

**Keywords:** fourier optics, apodization, bartlett window function, strehl-ratio, second-order moment

**INTRODUCTION**

The “Strehl ratio,” or “Strehl definition” is defined as the ratio of the central irradiance with the non-uniform pupil function, to that with the uniform pupil function. Strehl, after whose name this parameter for the assessment of an image quality is known in the literature, himself originally called it, “Definition Shelling Keit” (M.BORN and E. WOLF, 2006). In its original nomenclature, the term “definition” was used to mean ‘distinctness’ of an outline or ‘detail’ in the image. The Strehl ratio, abbreviated as \( SR \), is also known as the “Strehl intensity” or “definition brightness” or “Strehl criterion”. The efficiency of a non-Airy pupil is indicated by this parameter. It is obviously equal to unity for perfect systems without any aberration. Further The Strehl definition is very important in finding the degrading effects of aberrations of an incoherent optical image system, in which it is defined as the ratio of the light intensity at the maximum of the Point Spread Function (PSF) to the maximum of the \( PSF \) of the same system in the absence of aberrations. It can be also proved to be equal to the normalized volume under the optical Transfer Function (OTF) of an aberrated system. The Strehl ratio is sensitive to apodization, obscuration, defocusing, and image motion and wave-front error. However, for polychromatic \( PSF \) the Strehl ratio loses its relevance. Strehl ratio can also be computed from the Optical Transfer Function (OTF) of the system, which is again the Fourier transform of the \( PSF \). In the theory of image formation the pupil function and the distribution of amplitude in the image are related by means of a Fourier transformation. The technique of apodisation is widely used in the process of image restoration and image enhancement when the object is illuminated by incoherent light. In what follows in the next section, we shall present a brief review of studies made by various authors to reveal the great importance of Strehl ratio as an image-quality assessment parameter of an optical system. The “Second order moment” or simply the second–moment plays an important role in apodisation studies. The minimization of the second moment gives the maximum value of the central intensity for a given pupil function. The second moment, usually denoted by the symbol \( \Delta \), is strongly dependent on the distant feet of the diffracted field.

**Previous Studies on Strehl Ratio**

The Strehl ratio is an important image quality assessment parameter for optical systems. That is why its maximization by the use of amplitude filters has been attempted by several works for various purposes. (BARAKAT, Rand HOUSTON, A, 1963) in his study on solutions of Luneburg’s apodization problems investigated the Strehl ratio for both circular and slit apertures. It is not a physically measurable quantity in the strict sense of the word, but nevertheless, is a common measure of theoretical performance of the system (WILKINS, J.E Jr.1979) while solving the modified Luneburg apodization problems, discussed the Strehl ratio. (BARAKAT and HOUSTON, A, 1963) have studied the apodization problem of determining the diffraction pattern to have the largest possible Strehl ratio, for a rotationally symmetric optical system. (BARAKAT and HOUSTON A, 1963) computed Strehl ratio for an annular aperture possessing third-order and fifth-order spherical aberration.

It was (DEVILIS, J.B., 1965) who, in his study of comparisons of methods of evaluation, discussed the...
Strehl ratio and its relation to Marechal tolerance, (HOPKINS, H. H.1966) stated that for highly corrected optical systems, that is, those substantially satisfy the Rayleigh quarter-wave criterion, the Strehl ratio may be used as diffraction based criterion of image quality. Strehl ratio for circular apertures with a ring-shaped π-phase change, has been investigated by (ASAKURA and MISHINA, H. 1970). This work has been extended by (ASAKURA and NAGAI 1971) to modified annular and annular apertures and it has been found that the Strehl ratio is always reduced in comparison with that of a clear aperture as long as the semi-transparent and phase annular aperture is used.

(KUSAKAWA and OKUDAIRA.1972), in their study of Weiner apodization problems obtained pupil functions for different Strehl ratios. The relation between the minimum obtainable second-order moment and the pre-specified Strehl ratio has been discussed by them. (HAZRAL.N,1975), studied the problem of maximization of Strehl ratio for the more general case of partially space-coherent illumination. Hazra restated the criterion of “Maximization of Strehl ratios” as the criterion of “maximization of effective central illumination within a circle of infinitesimally small radius around the centre of the diffraction pattern”. The apodisation problem of finding the diffraction pattern has specified Sparrow limit of resolution and the maximum possible Strehl criterion has been solved by (PENG and WILKINS,J.E., Jr.1975), for both incoherent and coherent illumination respectively. (WILKINS, T.L.1973) solved the apodisation problem for maximum Strehl ratio and specified Rayleigh limit of resolution (STAMNES J.J.1981), while re-examining the Lunenburg apodisation problem in the frame-work of non-paraxial optics, concluded that a converging spherical wave with a uniform energy distribution as compared to a converging spherical wave with a uniform energy distribution over the aperture, always gives better results, so far as the Strehl ratio is concerned. (MAHAJAN,1973), calculated the Strehl ratio, quite accurately from the phase aberration variance. (KIBE and WILLIAMS J.E., Jr.1984) have studied Strehl ratio for a specified Rayleigh limit and for maximum central irradiance. (LOHMANN et al 1994) to derive the condition for axial symmetry and periodicity of Strehl ratio, which may serve as the best focus criterion.

Strehl ratio for the Straubel class of apodisation filters has been studied by (RAO,K.P et al 1977), concluded that the Strehl ratio is the encircled energy enclosed within a circle of infinitesimal radius. Strehl ratio for triangular and associated filters has been investigated by (VISHWANATHAM,1984) several others, like DEVARAYALU,etal,1979), have employed different apodisers and studied the effect on the Strehl ratio (HEROLOSKI R.1985) has derived enclosed form solutions for Strehl ratio of an untruncated and aberrated Gaussian beam system.

Formulae for estimating the Strehl co-efficient in the presence of third and fifth-order aberrations as well as defocusing have been obtained by (RAMANATHAN.S.et al, 1981), examined the effect of Kaiser pupils on the Strehl ratio. (MURTHY P.V.V.S. 1992) used co-sinusoidal filters and investigated the influence of apodisation and defocusing, with both circular and annular apertures on Strehl ratio. (SURENDER K.et al.1993), has evaluated the Strehl ratio for apodised optical systems, circular and annular, using Lanezo’s filters and determined that apodisation in combination with obscuration further lowers the Strehl ratio. (KARUNA SAGAR.A.2003) has evaluated the Strehl ratio for both circular and annular apertures apodised with generalized Hanning filters for the first, second, third and the forth orders of the filter functions. (VJENDER.C. et al 2013) have evaluated Strehl-Ratio and Second-Order moment with Co-Sinusoidal pupil function \( f(r) = \frac{1 + \beta \cos \pi r^2}{1 + \beta} \). In the present paper we have evaluated Strehl-Ratio and Second-Order moment with apodised Bartlett window function that is pupil function \( f(r) = (1 - \beta r) \), the results obtained have been discussed with the help of table and figures.

3. Mathematical Formula for Strehl ratio (SR) and Second order moment (\( \Delta \)):

The Strehl ratio denoted with SR, by definition the Strehl ratio can be mathematically defined as

\[
SR = \left[ \frac{G_F(0,0)}{G_A(0,0)} \right]^2
\]  

where the subscripts \( F \) and \( A \) refer to the non-uniform and the uniform pupils respectively has shown that the equation (3.1) can be expressed as

\[
SR = \left[ \frac{G_F(0,0)}{G_A(0,0)} \right]^2 = 4 \left[ \int_0^1 f(r) r dr \right]^2
\]  

The above expression for \( SR \) in terms of the pupil function \( f(r) \) has made it very easy to compute the Strehl ratio for a given optical system. (WYANT and CREATH.1992) redefined Strehl ratio, in terms of the wave front aberration \( \Delta W \), as

\[
SR = \frac{1}{\pi^2} \left[ \int_0^{2\pi} \int_0^{2\pi} e^{i2\pi \Delta w (r, \theta)} r dr d\theta \right]^2
\]  

where \( \Delta w \) in units of waves, is the aberration of the wave front , relative to the reference sphere for
diffraction focus. The equation (3.3) can be expressed in the form

$$SR = \frac{1}{\pi} \left[ \int_{0}^{\pi} \int_{0}^{\pi} \left(1 + \frac{1}{2}(1.2\pi \Delta w)^2 + \ldots \right) r dr d\theta \right]$$  \hspace{1cm} (3.4)

If the aberrations are so small that the third-order and higher order powers of $(2\pi \Delta w)$ can be neglected, equation (3.4) may be written as Strehl ratio

$$\approx 1 + i(2\pi \Delta w - \frac{1}{2}(2\pi)^2 \Delta w^2)^2$$

$$\approx 1 - (2\pi)^2 [\Delta w^2 - (\Delta W)^2] \approx 1 - (2\pi)^2$$  \hspace{1cm} (3.5)

In the above equation, $\Delta W$ and $\Delta W^2$ are the mean values of $\Delta w$ and $\Delta w^2$ respectively. $\sigma$ is the RMS value for the field-independent third-order aberrations $\sigma$ is expressed in units of waves. Thus, when the aberrations are small, the Strehl ratio is independent of the nature of the aberration and is smaller than the ideal value of unity by an amount proportional to the variance of the wavefront deformation. Further, the expression (3.5) is valid for Strehl ratios as low as about 0.5. Strehl ratio is always somewhat larger than would be predicted by the equation (3.5). (WYANT and CREATH. 1992) also showed that a better approximation for most types of aberration is given by

$$SR \approx e^{-(2\pi)^2} \approx 1 - (2\pi)^2 + \frac{(2\pi)^4}{24} + \ldots$$  \hspace{1cm} (3.6)

This is good for Strehl ratios as small as 0.1

Now, we have two equations, viz., (3.2) and (3.6) for computing the Strehl definition of an optical system. The former is applicable only to the diffraction-limited perfect Airy systems and the latter is to be used for aberrated systems. In the present paper, we are concerned with diffraction-free systems only. Therefore we have used the equation (3.2) for our computations.

The second moment is a measure of the flux concentration in the near vicinity of the axis in the diffraction pattern and has been expressed analytically as,

$$\Delta = \int_{0}^{\pi} \int_{0}^{\pi} G_x (0, Z) G_y (0, 0) Z^2 dZ$$  \hspace{1cm} (3.7)

(T.KUSAKAWA and J.OKUDAIRA. 1974) have expressed the second moment in terms of the pupil function as

$$\Delta = \left[ \int_{0}^{1} f'(r)^2 r dr \right]^2$$ \hspace{1cm} (3.8)

$$\int_{0}^{1} f(r) r dr$$

where $f'(r)$ is the first derivative of $f(r)$ with respect to $r$, in the case of the apodisation filters under our consideration in this paper. The expression for the second moment can be obtained by substituting $f(r) = (1 - \beta r)$ in the above equation. Thus, we get,

$$\Delta = \left[ \int_{0}^{1} \frac{d}{dr} (1 - \beta r)^2 r dr \right]^2$$ \hspace{1cm} (3.9)

$$\Delta = \left[ \int_{0}^{1} (1 - \beta r)^2 r dr \right]^2$$ \hspace{1cm} (3.10)

$$\Delta = \frac{\beta^2}{\left[ \int_{0}^{1} (1 - \beta r)^2 r dr \right]^2}$$ \hspace{1cm} (3.11)

In above equations for The Bartlett window functions the pupil function consider as below

$$f(r) = (1 - \beta r)$$  \hspace{1cm} (3.12)

Where $\beta$ is the apodisation parameter which controls the transmission over the exit pupil of the optical system and $r$ is the normalized distance of a point on the pupil from its centre.

**RESULTS AND DISCUSSIONS**

**Strehl Ratio $SR$** We have shown the computed values of the Strehl ratio for various values of $\beta$ in the table1 the variations of $SR$ with $\beta$ has also been shown in the figure1. From both the table and the graph it is evident that the value of $SR$ decreases with increase in the value of $\beta$ , the apodization parameter. This implies that the quality of the point image is degraded steadily as the value of $\beta$ is increased. Applying the Marechal criterion to the results obtained by us i.e., the tolerable value of $SR$ must not be less than 0.8, we find that, in order to obtain a good quality image of point object, by the system considered by us, the value of $\beta$ must be within the range $0 \leq \beta \leq 0.165$.

**Second order Moment $\Delta$** We have used the expression (5) to evaluate the second Order Moment $\Delta$. The Computed values of $\Delta$ for values of $\beta$ have been shown in the table2. The variation of $\Delta$ with $\beta$ have been shown graphically in the figure2. It found, from both the table and the figure, that the value of $\Delta$ Can be kept low only in the range
For increasing the values of $\beta$ thereafter is an advantage for the image quality.

CONCLUSIONS

The value of Strehl-Ratio decreases with increase in the value of the apodisation parameter Beta. For lower values of Beta in the range $0 \leq \beta \leq 0.3$ there is no improvement in the Second Order Moment it becomes very low, increasing values of Beta in the range $0.4 \leq \beta \leq 1$ the Second-Order Moment increased very rapidly.

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REFERENCES


Murty, P.V.V.S, (2003).Thesis entitled “Studies on the Diffacted field and Imaging Characteristics of optical systems with Co-sinusoidal apodisation filters” presented to Osmania University for Ph.D.


Fig: 1. Variation of SR with Beta.

Fig: 2. Variation of Second order Moment with Beta.

Table 1: Strehl ratio (SR) values for various values of Beta

<table>
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<tr>
<th>Beta</th>
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<td>1.0</td>
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Table 2: Second order moment values for various values of Beta

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<th>Δ values</th>
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