Numerical Schemes Based on Non-Standard Methods for Initial Value Problems in Ordinary Differential Equations

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Abstract
In this paper we present new methods of constructing Non-Standard schemes. Two new logical ways for constructing the denominator functions for the discrete derivatives were introduced and schemes were constructed based on Non Local Approximation. The schemes were applied on sampled IVPs in ordinary differential equations to verify whether the schemes are computationally reliable. The results obtained show that the schemes are suitable for problems for which they were proposed.

Keywords: local approximations, denominator functions (type A and B), modeling, non-standard method, IVP (Initial Value Problem), rock.

INTRODUCTION
In this paper we developed computationally reliable Non-Standard difference schemes that support the qualitative properties of the considered initial value problems. Standard Finite difference schemes for differential equations exhibit a level of numerical instability (Mickens 1994).

Mickens (1994, 2000), gave valuable reasons for numerical instabilities in some particular investigated cases. Thus, the preservation of the qualitative properties of the considered differential equation with respect to these schemes is of great significance. He proposed a new method of construction of discrete models whose solution have the same qualitative properties as that of the corresponding differential equation for all step size and thus eliminate the elementary numerical instabilities that can arise in finite difference models of differential equations. The major consequence of this result is that such scheme does not allow numerical instabilities to occur (Mickens 1994).

Anguelov and Lubuma (2003) based their works on Mickens (1994, 2000) and deduced a set of non-standard modeling techniques using the two rules given below
i. The denominator can be replaced by a function \( \varphi(h) = h + o(h^2) \) as \( h \to 0 \) and \( o(\varphi(h)) < 1, h \to 0 \) which can be considered as the renormalization of step size \( h \).
ii. That the non-linear terms must be approximated non-locally on the computational grid by a suitable function of several points of the mesh

Other works of Ibijola and Sunday (2010), Sunday, Ibijola and Skwame (2011), discussed extensively on Exact Non-standard schemes while further works of Ibijola, Lubuma and Ade-Ibijola 2010 present the theory of nonstandard finite difference schemes which can be used to solve some initial value problems in ordinary differential equations. Methods of construction and mode of implementation of these methods were also discussed. The motivation for these work is that of Anguelov & Lubuma (2003) and Ibijola & Sunday (2010).

Throughout this work we shall consider problems of the form

\[ \frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0 \quad (1) \]

Which occur most often in Biological, Chemical, Physical, Management sciences and Engineering.

The aim of this research work is to construct discrete models whose solution have the same qualitative properties as that of the corresponding differential equation for all step size and thus eliminate the elementary numerical instabilities that can arise in finite difference models.

Exact Finite-Difference Scheme

Definition 1 (Mickens 1994)
An exact finite-difference scheme is one which the solution to the difference equation has the same general solution as the associated differential equation.

Definition 2: Type A denominators (increment functions) are those constructed with the knowledge of analytic solution. Type B denominators are those constructed otherwise.

We will exploit the point of intersection between the exact finite difference schemes and the expected
solution of the original differential equation and use Mickens’ rules [Mickens (1994)] to develop new Non-Standard finite difference schemes.

**Non-Standard Finite Difference Modeling Rules**

A finite difference scheme is called non-standard finite difference method, if at least one of the following conditions is met (Anguelov and Lubuma 2003):

i. In the discrete derivative, the traditional denominator is replaced by a non-negative function \( \varphi \) such that, \( \varphi (h) = h + o(h^2) \), as \( h \) 0.

ii. Non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable function of several points of the mesh.

**Derivation of the New Method**

The discrete solution of a finite difference scheme can be written as

\[ y_{n+1} = \alpha y_n \]  

where, 

\[ \alpha = g(y(x_{n+1}), y(x_n)) \]  

\( \alpha \) is a real valued function. A suitable candidate for the function \( g \) is

\[ y(x_k) \text{ is the analytic solution at } x_k. \]

i. Consider the non-standard finite difference scheme below for any dimensionless equation of the form (1.0) given by

\[ \frac{y(x_{n+1}) - y(x_n)}{\varphi(h)} = f(x_n, y_n) \]  

or

\[ y_{n+1} = y_n + \varphi f(x_n, y_n) \]  

\[ \text{or} \quad \varphi = \frac{\varphi f(x_n, y_n)}{y(x_{n+1}) - y(x_n)} \]  

ii. Consider also the exact scheme in the form

\[ y_{n+1} = \frac{y(x_{n+1})}{y(x_n)} \]  

Take

\[ \alpha = \frac{\varphi}{y(x_{n+1}) - y(x_n)} = g(y(x_{n+1}), y(x_n)) \]

If we compare these schemes in (i) and scheme (ii) above

\[ y_{n+1} = y_n + \frac{\varphi f(x_n, y_n)}{\varphi} = f(x_n, y_n) \]

Then \( \varphi \) can be chosen as

\[ \varphi = \frac{y(x_{n+1})}{y(x_n)} - 1 \]

\[ \frac{\varphi f(x_n, y_n)}{f(x_n, y_n)} \]

Hence, we have a method of choosing \( \varphi \):

\[ y_{n+1} = y_n + \varphi f(x_n, y_n) \]

\[ \varphi = \frac{y(x_{n+1})}{y(x_n)} - 1 \]

\[ \frac{\varphi f(x_n, y_n)}{f(x_n, y_n)} \]

We can consider the point where the expected analytical and the numerical solution coincides i.e.

\[ y(x_k) = y_k \]  

then the scheme in (5) can be written as:

\[ \varphi = \frac{y(x_{k+1}) - y(x_k)}{f(x_k, y_k)} \times \frac{1}{f(x_k, y_k)} \]  

(6)

Then taking limit as \( h \) approaches zero

\[ \varphi = \frac{y(x_{k+1}) - y(x_k)}{h} \times \frac{h}{f(x_k, y_k)} \]

(7)

Where \( f'(x, y) \) is the derivative of \( f(x, y) \) and \( f'(x_k, y_k) \) is the value \( f(x, y) \) at point \( x_k \)

\( h \) may be replaced with higher powers of \( h \) to have \( \varphi \) \( = h + o(h^2) \) as \( h \) 0 (Mickens 1994)

Using the following normalization (\( \varphi \) ) functions we will test behavior of each scheme to be developed:

1) \( \varphi = \sin h \)

2) \( \varphi = 1 - e^{-h} \)

3) \( y_{k+1} = \alpha y_k \) \( \alpha = \frac{y(x_{k+1})}{y(x_k)} \) exact scheme

4) \( \varphi = \frac{\varphi - 1}{y(x_k)} \) Derived from Analytic solution

5) \( \varphi = h f(x, y) \) Derived from taking derivative

**Construction of the New Non Standard Schemes**

**Problem 1:** Consider the IVP \( y^1 = 3y; \ y(0) = 50; \)

with exact solution \( y(t) = 50e^{3t} \)  

a) At the points \( x = x_{k+1} \) and \( x = x_k = kh; \)

\( y_{k+1} = 50e^{3(k+1)}h \) and \( y_k = 50e^{3(k)}h \) Let \( \alpha = \frac{y(x_{k+1})}{y(x_k)} \)

Then, the exact finite differences scheme is

\[ y_{k+1} = \alpha y_k \]

\[ (e^{3(k+1)}h / e^{3(k)}h) y_k \]  

(9)
b) Using \( 3y = 3(a y_k + (1 - a) y_{k+1}) \) a \( \in \mathbb{R} \) and (3) in (8) we obtain 
\[
Y_{k+1} - Y_k = 3\varphi \{a y_k + (1 - a) y_{k+1}\}
\]
(10)

Then 
\[
Y_{k+1} = \left(\frac{1 + 3a\varphi}{1 - 3\varphi + 3a\varphi}\right) Y_k
\]
(11)

Comparing this with the exact scheme \( Y_{k+1} = \alpha Y_k \) we have
\[
Y_{k+1} = \alpha Y_k = \left(\frac{1 + 3a\varphi}{1 - 3\varphi + 3a\varphi}\right) Y_k \quad \text{and} \quad \varphi = \frac{1 - \alpha}{3(a\alpha - \alpha - a)}
\]
(12)

c) Using (5) \( \varphi = \frac{(\alpha - 1)y_n}{f(x_n, y_n)} \)
we have
\[
\varphi = \frac{(\alpha - 1)y_n}{3y}
\]
(13)

d) Using (7) \( \varphi = h f'(x_n, y_n) \) we have
\[
f'(x_n, y_n) = 3 \varphi = h \left(\frac{3}{3y}\right)
\]
(14)

Or \( \varphi = h \) to protect the fact that \( \varphi (h) = h + o(h^3) \) as \( h \to 0 \)

e) Using (14) in \( Y_{n+1} = Y_n + \varphi f(x_n, y_n) \)
Then
\[
Y_{k+1} = 3h + Y_k
\]
(15)

**Problem 2**

Consider the IVP
\[
y^1 = y - 1 \quad y(0) = 2
\]
(16)

with analytic solution \( y(x) = 1 + e^x \)

At the points
\[
x = x_k = hk \quad \text{and} \quad x = x_{k+1} = (k + 1)h
\]
\[
y_k = 1 + e^{hk}, \quad y_{k+1} = 1 + e^{(k+1)h}
\]
(17)

The exact finite difference scheme is therefore given by
\[
Y_{k+1} = \frac{1 + e^{(k+1)h}}{1 + e^{kh}} Y_k
\]
where \( \alpha = \frac{1 + e^{(k+1)h}}{1 + e^{kh}} \)
(18)

a) Let \( \gamma = a y_n + (1 - a) y_{n+1} \) then from (3) and (16)

We obtain
\[
\gamma_n - y_n = a y_n + (1 - a) Y_n + 1
\]

Then a non-standard scheme is
\[
Y_{n+1} = \left(\frac{1 + 1000e^{0.2k}}{1 + 1000e^{0.2k}}\right) Y_k
\]
(19)

From (2) and (19) \( \alpha Y_n = \left(\frac{1 + a\varphi h}{1 + a\varphi h - a\varphi h}\right) y_{n+1} \)

\[
\varphi_h = \frac{(1 - \alpha)y_n}{a\alpha - \alpha - a} y_{n+1}
\]
Hence a non-standard scheme

\[
y_{n+1} = \frac{(1 + a\varphi h) y_n - a\varphi h}{(1 + a\varphi h - a\varphi h)} y_{n+1}
\]
(20)

b) Using (5) \( \varphi = \frac{(\alpha - 1)y_n}{f(x_n, y_n)} = \frac{(\alpha - 1)y}{y - 1} \)
(21)

Using (7)
\[
\varphi = h f'(x_n, y_n) = 1
\]
(22)

Or \( \varphi = h \) to protect the fact that \( \varphi (h) = h + o(h^3) \) as \( h \to 0 \)

e) Using (28) in \( Y_{n+1} = y_n + \varphi f(x_n, y_n) \)
Then
\[
Y_{k+1} = 0.2h + y_k
\]
(23)

**Problem 4**

Consider the IVP \( y' = \lambda y(1000 - y) \), \( \lambda = 1 \), \( y(0) = 100 \)
(24)

This is a form of logistic equation, the analytic solution is given by
\[
y(t) = \frac{1000}{1 + 9e^{-1000t}}
\]
(25)
and the Exact finite difference scheme is given by
\[ y_{k+1} = \alpha y_k \] (31)
Where
\[ \alpha = \left[ 1 + 9e^{-1000kh} \right]^{-1} \]
\[ 1 + 9e^{-1000(k+1)h} \]

a) Let \( y^2 = ay_k^2 + (1-a)y_ky_{k+1} \)
Then \( 1000y - y^2 = 1000y_k - (ay_k^2 + (1-a)y_ky_{k+1}) \)
\[ \Rightarrow y_{k+1} = \frac{y_k + 1000\varphi y_k - a\varphi y_k^2}{(1+\alpha a)\varphi y_k} \] (32)
From \( y_k + 1000\varphi y_k - a\varphi y_k^2 \)
Then the new scheme becomes
\[ y_{k+1} = \frac{y_k + 1000\varphi y_k - a\varphi y_k^2}{(1+\alpha a)\varphi y_k}, \quad \varphi = \frac{(\alpha-1)}{(1000 - (a + \alpha (1-a))y_k)} \] (33)
b) Using (5) \( \varphi = \frac{(\alpha-1)y_n}{f(x_ny_n)} = \frac{(\alpha-1)}{1000-y} \) (34)
c) Using(7) \( f'(x_ny_n) = 1000 - 2y \)
\[ \varphi = \frac{h(f'(x_ny_n))}{y(1000-y)} \]
\[ \alpha y_k \]
\[ (1000 - (a + \alpha a)y_k^2) \]
Then the new scheme becomes
\[ y_{k+1} = \frac{y_k + 1000\varphi y_k - a\varphi y_k^2}{(1+\alpha a)\varphi y_k} \] (36)
\[ \quad \varphi = \frac{(\alpha-1)y_n}{f(x_ny_n)} = \frac{(\alpha-1)}{1000-y} \] (38)

**Graphical Representation of the Numerical Result**
\[ y' = 3y, \quad y(0) = 50 \quad h = .01, \quad a = 0 \quad \phi = \frac{h}{y} \]

**Fig 1**
\[ \begin{align*}
\text{Fig 1} & \quad y' = 3y, \quad y(0) = 50 \quad h = .01, \quad a = 0, \quad \phi = \frac{h}{y} \\
\text{Fig 2} & \quad y' = y - 1, \quad y(0) = 2 \quad a = .05, \quad h = 0.1, \quad \phi = \frac{h}{y-1} \\
\text{Fig 3} & \quad y' = y - 1, \quad y(0) = 2 \quad a = .05, \quad h = 0.1, \quad \phi = \frac{h}{y} 
\end{align*} \]
Fig 4
\[ y' = 0.2y, \quad y(0) = 1000 \quad h = .01 \quad a = 0.5 \]
\[ \text{Phi} = h/y \]

Fig 6
\[ y' = 0.2y, \quad y(0) = 1000 \quad h = .01 \quad a = 0.5 \quad \text{phi} = h \]

Fig 7
\[ y' = y(1000 - y), \quad y(0) = 100; \quad \text{Group 1} \]
Scheme a = 0.5 h = 0.01 \( \Phi = h \left( \frac{1000 - 2y}{y(1000 - y)} \right) \)

Fig 8
\[ y' = y(1000 - y), y(0) = 100; \quad \text{Group 1} \]
Scheme a = 0.5 h = 0.01 \( \Phi = h \left( \frac{1000 - 2y}{y(1000 - y)} \right) \)

Fig 9
\[ y' = y(1000 - y), y(0) = 100; \quad \text{Group 2} \]
Schemes a = 0.5 h = 0.01 \( \Phi = \frac{h(1000 - 2y)}{y(1000 - y)} \)
\( y' = y(1000 - y), y(0) = 100; \quad \text{Group 2 Schemes} \quad a = 0.5 \ h = 0.1 \ \phi = h \)

![Graph of Group 2 Schemes](image)

**SUMMARY AND CONCLUSION**

We have been able to construct two groups or family of schemes for the last three IVP’s and one group each for the first two. The schemes have been tested numerically in terms of their consistency with the known behavior of the analytic solution of the particular initial value problem. The last equation is a form of the logistic equation. The Non-standard schemes developed are not consistent with the analytic solution for a > 0. It can be seen that the schemes oscillate close to the fixed point \( y = 1000 \). The result is expected to be different if the equation is represented in its dimensionless form. The methods based on type B denominators will behave better if it is modified to satisfy the non-standard modeling rules. This is because the \( \phi \) tends to produce a linear scheme most especially when they are applied directly to the non-standard Finite difference scheme(3). The advantage of the last technique is that the analytic solution does not have to be known provided it exist. The construction of \( \phi \) to satisfy the Mickens rules is sacrosanct to the validity of this new techniques. An area of improvement is to develop a method of modifying \( \phi \) accordingly to satisfy the requirement of Non-standard schemes. The numerical experiment has shown that: The denominator function can be constructed in the following ways (in the tested cases).

\[
\phi = \sin(h), 1 - e^{-h} \quad \text{i.e using classical functions (Mickens 1994)}
\]

\[
\phi = \frac{(a-1)y_k}{f(x,y)}, \quad a = g \left[ y(x_{n+1}), y(x_n) \right], \quad y(x) \text{is the known analytic solution.}
\]

\[
\phi = \frac{h^n f^2(x,y)}{f(x,y)} \quad n \geq 1.
\]

All the schemes have been found to be consistent with literature and compared favorably in a many cases. However the schemes (35 and 39) do not exhibit the same qualitative property of the corresponding IVP. Even where \( \phi(\phi) \) is an exponential or sine function.

**REFERENCES**


