Formulation of the Equilibrium Equations of Transversely Loaded Elements Taking Beam-Column Effect into Consideration

Okonkwo V. O and Onyeyili I. O

Department of Civil Engineering
Nnamdi Azikiwe University, Awka

Corresponding Author: Okonkwo V. O

Abstract
In this work a mathematical model for the consideration of beam-column effect in structural analysis was formulated. The stiffness matrix for a prismatic element taking beam-column effect into account was developed. The force/load vectors for various cases of transversely loaded elements taking beam-column effect into consideration were also formulated and these were presented in tables synonymous to the tables of ‘end forces due to unit end displacements of prismatic elements’ and ‘fixed end moments on transversely loaded elements’ found in many structural analysis textbooks. In the analysis of structures by the stiffness method there is need to obtain the fixed end moments of the transversely loaded elements and also the end stiffness (forces due to unit end displacement of the elements) of these elements. These are evaluated using formulas obtained from textbooks. Unfortunately in the derivation of such formulas beam-column effect was ignored. In this work similar tables that incorporate beam-column effects were developed enabling an easy implementation of the effects of axial forces on the end stiffnesses of elements in structural analysis.

Keywords: stiffness, beam-column effect, degrees of freedom, prismatic members, modulus of elasticity

INTRODUCTION
When an axial force is present in a member, two effects are noticeable. One is a change in geometry arising from a change in length of members and the second is a change in stiffness of the member arising from bending by the axial force. The first effect is known as axial deformation. It is the only deformation used for the generation of the compatitibility equations used in the analysis of indeterminate trusses (Hibbeler, 2006). The second effect also known as the beam column effect (McGuire et al, 2000) increases or reduces the forces required to cause a unit rotation or translation at the end of a member. The force decrease (the stiffness of the member decrease) if the member is subjected to an axial compressive force and conversely increase (the stiffness of the member increase) if the axial force is tensile (Ghali and Neville, 1996).

MODEL
The analysis of structures by the stiffness method involves the writing of equilibrium equations for the degrees of freedom (coordinates) of the structure (Jenkins, 1990). The equilibrium equation for the analysis of an element is given by \([k]\{d\} = \{q\}\) (1)
Where \([k]\) is the element stiffness matrix, it is a 12 x 12 matrix for a space element (elements that can deform in all three coordinate axes) and a 6 x 6 for a plane element (elements that deform in only one plane). \(\{q\}\) is the vector of external forces applied at any of the nodes and which coincide with one of the degrees of freedom for which the equilibrium equations were written. \(\{d\}\) is the vector of displacements at the coordinates or degrees of freedom of the element.

When there is no external force on any of the degrees of freedom or coordinates equation (1) is rewritten as
\([k]\{d\} = 0\) (2)
But for transversely loaded elements equation (1) is written as \(\{q_o\} + [k]\{d\} = \{q\}\) (3)
(Leet and Unang, 2002)
where \(\{q_o\}\) is the vector of reactive end forces on the transversely loaded element when displacements at its coordinates (degrees of freedom) are restrained.

End Forces on Transversely Loaded Elements
Here it will be illustrative to represent each coordinate (degree of freedom) with a number. This is shown in Figure 1(a) below. Figure 1(a) & (b) shows the 12 degrees of freedom of a space element and their representation with numbers. If the plane element is lying in the xy plane (see Figure 2) and all the external loads on it act in the same plane then
\(d_3 = d_4 = d_5 = d_6 = d_{10} = d_{11} = 0\)
If the deformation in these coordinates is zero then the forces (or moments for rotations) in these coordinates will also be zero.
\(F_3 = F_4 = F_5 = F_9 = F_{10} = F_{11} = 0\)
\[ q_0 = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \ell \\ F_{12} \end{bmatrix} \]
\[
[k] = \begin{bmatrix}
 k_{11} & k_{12} & k_{16} & k_{17} & k_{18} & k_{112} \\
 k_{21} & k_{22} & k_{26} & k_{27} & k_{28} & k_{212} \\
 k_{61} & k_{62} & k_{66} & k_{67} & k_{68} & k_{612} \\
 k_{71} & k_{72} & k_{76} & k_{77} & k_{78} & k_{712} \\
 k_{81} & k_{82} & k_{86} & k_{87} & k_{88} & k_{812} \\
 k_{121} & k_{122} & k_{126} & k_{127} & k_{128} & k_{1121} 
\end{bmatrix}
\]

These are the Euler's differential equation governing the deflection of a transversely loaded prismatic member subjected to a compressive force \( P \) is
\[
\frac{d^2y}{dx^2} + u^2 \frac{d^2y}{dx^2} = q
\]

The solution of the differential equation is
\[
y''(x) = A_1 + A_2 x + A_3 \sin ux + A_4 \cos ux + \frac{q x^2}{p^2}
\]

The first derivative of equation (7) gives the slope
\[
y'''(x) = A_2 + u A_3 \sin ux - u A_4 \cos ux + \frac{q x}{p^2}
\]

But \( M(x) = -EIy''(x) \)

\[
M(x) = EIu^2 A_3 \sin ux + EIu^2 A_4 \cos ux - \frac{Elq x}{p}
\]

Substituting equation (19) and (21) into equation (18) will give
\[
Q(x) = -EIu^2 A_2 - \frac{Elq x}{p} \left(1 + \frac{u^2 x^2}{2}\right)
\]

Let the initial values of these parameters (ie values at \( x = 0 \)) be \( y_0, \theta_0, M_0 \) and \( Q_0 \), respectively. By applying these initial conditions into these general solution of the equation of equilibrium of the axially loaded uniform column (equations 18, 19, 20 and 22) the arbitrary constants \( A_1 - A_4 \) are obtained as
\[
y_o = \frac{E I u^2}{M_o} A_2 = \frac{E I u^2}{M_o} A_2 = \frac{E I u^2}{M_o} A_2 = \frac{E I u^2}{M_o} A_2 + \frac{q}{P u^2}
\]

These are substituted into equations (7), (8) and (11) and rearranged to obtain
\[
y''(x) = y_0 + \frac{\theta_0 \sin ux}{u} + \frac{M_o \cos ux - 1}{E I u^2} + \frac{Q_o}{E I u^2} + \frac{q}{P u^2} \left( \cos ux - 1 + \frac{u^2 x^2}{2} \right)
\]
\[
\theta(x) = \theta_0 \cos ux - M_o \frac{\sin ux}{E I u^2} + \frac{Q_o}{E I u^2} + \frac{q}{P u} \left( \sin ux - \frac{u x^2}{2} \right)
\]
same can be written by replacing \( P \) with \(-P\) so that \( u \) is replaced with \( iu \) where \( i = \sqrt{-1} \).

if \( \beta = uL \) then

\[
\begin{align*}
\sin i\beta &= i\sin h\beta \\
\cos i\beta &= \cos \beta \\
(i\beta)^2 &= -\beta^2 \\
(i\beta)^3 &= -i\beta^2
\end{align*}
\]

(Stroud, 1995; Duffy, 1998)

From the calculations above the table of fixed end forces of transversely loaded beam columns was obtained and presented as table 1 below (See Appendix).

Note that in the table above clockwise moments were taken as positive while anticlockwise moments were negative. In the derivation of the end moments and forces moments that keep the bottom fibres in tension were taken as positive while those that keep the top fibres in tension were negative. Likewise shearing forces that cause the system to move upwards were treated as positive in the table. In the derivation, shearing forces that cause the system to rotate clockwise were positive while the opposite were negative.

**End Forces due to End Displacements of Elements**

The Euler’s differential equation governing the deflection of a prismatic member (without transverse loads) subjected to a compressive force \( P \) is

\[
d^4y/dx^4 + u^2 d^2y/dx^2 = 0
\]

Where \( u^2 = \frac{P}{EI} \) and \( P \) is the compressive force.

The solution of the differential equation is

\[
y'(x) = A_1 + A_2 x + A_3 \cos ux + A_4 \sin ux
\]

The first derivative of equation (39) gives the slope

\[
y''(x) = \theta_o - u A_3 \sin ux + u A_4 \cos ux
\]

\[
y'''(x) = -u^2 A_3 \cos u x - u^2 A_4 \sin u x
\]

But \( M_{(x)} = -EI y''(x) \)

\[
M_{(x)} = EI u^2 A_3 \cos ux + EI u^2 A_4 \sin ux
\]

By substituting equation (21) and (30) into equation (16) we obtain

\[
Q_{(x)} = -EI u^2 A_2
\]

Let the initial values of these parameters (ie values at \( x = 0 \)) be \( y_o, \theta_o, M_o \) and \( Q_o \) respectively. By applying these initial conditions into these general solution of the equation of equilibrium of the axially loaded uniform columns, equations (29, 30, 31 and 32) the arbitrary constants \( A_1 - A_4 \) are obtained as

\[
\begin{align*}
y_o &= \frac{M_o}{EI u^2}, \\
A_2 &= \frac{M_o}{EI u^2}, \\
A_3 &= \frac{M_o}{EI u^2}, \quad \text{and} \quad A_4 = \frac{\theta_o}{u} + \frac{Q_o}{EI u^3}
\end{align*}
\]

Substituted into equations (29) – (32) and rearranged to obtain

\[
\begin{align*}
y_{(x)} &= y_o + \theta_o \sin ux \quad + \frac{M_o \cos ux - 1}{EI u^2} + \\
Q_o \sin ux - ux \quad + \frac{M_o \cos ux - 1}{EI u^2} \quad (33)
\end{align*}
\]

\[
\begin{align*}
\theta_{(x)} &= \theta_o \cos ux - \frac{M_o \sin ux}{EI u} + \frac{Q_o \cos ux - 1}{EI u^2} \quad (34)
\end{align*}
\]

\[
\begin{align*}
M_{(x)} &= \theta_o E l u \sin ux + \frac{M_o \cos ux + Q_o}{EI u^2} \quad (35)
\end{align*}
\]

\[
Q_{(x)} = Q_o \quad (36)
\]

Equations (33) – (36) were used to obtain the stiffness coefficients of axially loaded uniform elements due to unit end displacements and is presented in table 2.

The stiffness of a prismatic member in tension were derived by replacing \( P \) with \(-P\) so that \( u \) is replaced with \( iu \), \( \beta \) with \( i\beta \) where \( i = \sqrt{-1} \).

Note that in the table above clockwise moments were taken as positive while anticlockwise moments were negative. In the derivation of the end moments and forces moments that keep the bottom fibres in tension were taken as positive while those that keep the top fibres in tension were negative. Likewise shearing forces that cause the system to move upwards were treated as positive in the table. In the derivation, shearing forces that cause the system to rotate clockwise were positive while the opposite were negative.

**SUMMARY AND CONCLUSION**

Beam column effect is often ignored in structural analysis. It is implemented mostly in non-linear analysis by commercial software. The development of the ‘fixed end moments on transversely loaded prismatic elements’ and ‘The end forces due to unit end displacements of prismatic elements’ which are presented in tables 1 and 2 would facilitate an easy implementation of beam column effect in structural analysis of frames. It is important to note that the use of tables 1 and 2 requires the knowledge of the axial force in the element. Hence the frame has to be analysed first to determine the forces in the members before the effects of beam column can be considered. When beam column effect is ignored i.e the axial force in the element \( P \) is taken to be zero in the calculation the element stiffness and load vectors, tables 1 and 2 will give the same values as obtained from similar tables in structural engineering textbooks like Reynolds and Steedman (2001) and Davison and Owens (2007).

**REFERENCES**


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APPENDIX

Table 1: End forces on transversely loaded prismatic members under axial loads

<table>
<thead>
<tr>
<th>S/N</th>
<th>Beam</th>
<th>Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Diagram 1" /></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_1 = \frac{qL^2}{\beta^2} \left(1 - \frac{1}{2} \left(1 + \cos \beta \right) \frac{1}{\sin \beta} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_2 = \frac{qL^2}{\beta^2} \left(1 - \frac{1}{2} \left(1 + \cos \beta \right) \frac{1}{\sin \beta} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$F_1 = F_2 = \frac{qL}{2}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td><img src="image2" alt="Diagram 2" /></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M = -\frac{qL^2}{2\beta^2} \left(2\cos \beta - 2 + \beta^2 \right) \left(\sin \beta - 2(\sin \beta - \beta)(\cos \beta - 1)\right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$F_1 = \frac{qL}{2\beta} \left(\sin \beta (\cos \beta - 1) - \cos \beta (\sin \beta - \beta)\right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$F_2 = \frac{qL}{2\beta} \left(2\cos \beta + \beta^2 \cos \beta - 2\right)$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td><img src="image3" alt="Diagram 3" /></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_1 = -\frac{qL^2}{6\beta^3} \left(\cos \beta - 1\right)(6\sin \beta - 6\beta + \beta^3) - 3(\sin \beta - \beta)(2\cos \beta - 2 + \beta^2) \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_2 = \frac{2kL^3 \cos \beta}{6\beta^3}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$F_1 = \frac{qL}{6\beta^2} \left(\sin \beta (6\beta - 6\sin \beta - \beta^3) - 3(\cos \beta - 1)(2\cos \beta - 2 + \beta^2)\right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$F_2 = \frac{qL}{6\beta^2} \left(2 - 2\cos \beta - \beta \sin \beta \right)$</td>
<td></td>
</tr>
</tbody>
</table>

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\[ M = -qL^2 \left[ \frac{6\beta \sin \beta + \beta^3 \sin \beta - 6\beta^2}{\beta \cos \beta - \sin \beta} \right] \]
\[ F_1 = \frac{qL}{6\beta^2} \left[ \frac{\beta^3 \cos \beta + 6 \sin \beta - 6\beta}{\beta \cos \beta - \sin \beta} \right] \]
\[ F_2 = \frac{qL}{6\beta^2} \left[ \frac{\beta^3 \cos \beta + 6 \sin \beta - 6\beta}{\beta \cos \beta - \sin \beta} - 3\beta^2 \right] \]

\[ M = \frac{qL^2 \sin \beta}{6\beta^2} \left[ \frac{2\beta^3}{\beta \cos \beta - \sin \beta} + 6\beta \sin \beta \right] \]
\[ F_1 = \frac{qL}{6\beta^2} \left[ \frac{-6 + \beta^3 \cos \beta - 3\beta^2 \sin \beta}{\beta \cos \beta - \sin \beta} \right] \]
\[ F_2 = -\frac{qL}{6\beta^2} \left[ \frac{-6 + \beta^3 \cos \beta - 3\beta^2 \sin \beta}{\beta \cos \beta - \sin \beta} - 3\beta^2 \right] \]

\[ M_1 = -qL^2 \left[ \frac{1 + \cosh \beta}{\beta^2 \sinh \beta} \right] - 1 \]
\[ M_2 = \frac{qL^2}{\beta^2} \left[ \frac{1 + \cosh \beta}{\sinh \beta} \right] - 1 \]
\[ F_1 = F_2 = \frac{qL}{2} \]

\[ M = \frac{qL^2}{2\beta^2} \left[ \frac{2(\sinh \beta - \beta)(\cosh \beta - 1) - \sinh \beta (2 \cosh \beta - 2 + \beta^2)}{\sinh \beta (\cosh \beta - 1) - \cosh \beta (\sinh \beta - \beta)} \right] \]
\[ F_1 = \frac{qL}{2\beta} \left[ \frac{\sinh \beta (\cosh \beta - 1) - \cosh \beta (\sinh \beta - \beta)}{2 + \beta^2 \cosh \beta - 2 \cosh \beta} \right] \]
\[ F_2 = -\frac{qL}{2\beta} \left[ \frac{\sinh \beta (\cosh \beta - 1) - \cosh \beta (\sinh \beta - \beta)}{2 + \beta^2 \cosh \beta - 2 \cosh \beta} - 2\beta \right] \]

\[ M_1 = -\frac{qL^2}{6\beta^3} \times \left[ 3(\sinh \beta - \beta)(2 \cosh \beta - 2 - \beta^2) - (\cosh \beta - 1)(6 \sinh \beta - 6\beta - \beta^3) \right] \]
\[ M_2 \times \left[ 2 - 2 \cosh \beta + \beta \sinh \beta \right] \]
\[ M_2 \times \left[ 2 - 2 \cosh \beta + \beta \sinh \beta \right] \]
\[ M_2 \times \left[ 2 - 2 \cosh \beta + \beta \sinh \beta \right] \]
\[ M \times \left[ 2 - 2 \cosh \beta + \beta \sinh \beta \right] \]
\[ \frac{qL}{P} \]
\[ F_1 = \frac{qL}{6\beta^2} \times \left[ \sinh \beta (6\beta - 6 \sinh \beta + \beta^3) - 3(\cosh \beta - 1)(2 \cosh \beta - 2 + \beta^2) \right] \]
\[ F_2 = -\frac{qL}{6\beta^2} \times \left[ \sinh \beta (6\beta - 6 \sinh \beta + \beta^3) - 3(\cosh \beta - 1)(2 \cosh \beta - 2 + \beta^2) \right] \]
\[ F_2 = -\frac{qL}{6\beta^2} \times \left[ \sinh \beta (6\beta - 6 \sinh \beta - \beta^3) - 3(\cosh \beta - 1)(2 \cosh \beta - 2 + \beta^2) \right] \]
\[ -3\beta^2 \]
Where $\beta = uL$ and $u = \sqrt{\frac{P}{EI}}$

Table 2: End forces caused by end displacement of prismatic members putting beam column effects into consideration.

<table>
<thead>
<tr>
<th>S/No</th>
<th>Beam</th>
<th>Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="#" alt="Beam Diagram" /></td>
<td>$F_1 = -F_2 = \frac{EIL^3}{2 - 2\cos \beta - \beta \sin \beta}$, $M_1 = M_2 = \frac{EIL^2}{2 - 2\cos \beta - \beta \sin \beta}$</td>
</tr>
<tr>
<td>2</td>
<td><img src="#" alt="Beam Diagram" /></td>
<td>$F_1 = -F_2 = \frac{EIL^2}{2 - 2\cos \beta - \beta \sin \beta}$, $M_1 = M_2 = \frac{EIL}{2 - 2\cos \beta - \beta \sin \beta}$</td>
</tr>
<tr>
<td>3</td>
<td><img src="#" alt="Beam Diagram" /></td>
<td>$F_1 = -F_2 = \frac{EIL^3}{\sin \beta - \beta \cos \beta}$, $M_1 = M_2 = \frac{EIL^2}{\sin \beta - \beta \cos \beta}$</td>
</tr>
<tr>
<td>4</td>
<td><img src="#" alt="Beam Diagram" /></td>
<td>$F_1 = -F_2 = \frac{EIL^2}{\sin \beta - \beta \cos \beta}$, $M_1 = M_2 = \frac{EIL}{\sin \beta - \beta \cos \beta}$</td>
</tr>
<tr>
<td>5</td>
<td><img src="#" alt="Beam Diagram" /></td>
<td>$F_1 = -F_2 = \frac{EIL^3}{2 - 2\cos \beta + \beta \sin \beta}$</td>
</tr>
</tbody>
</table>
\[ M_1 = M_2 = \frac{E_1 \beta^2}{L^2} \left( \frac{\cos h \beta - 1}{2 - 2 \cos h \beta + \beta \sin h \beta} \right) \]

\[ F_1 = -F_2 = -\frac{E_1 \beta^2}{L^3} \left( \frac{\cos h \beta}{\beta \cos h \beta - \sin h \beta} \right) \]

\[ M = -\frac{E_1 \beta^2}{L^2} \left( \frac{\sin h \beta}{\beta \cos h \beta - \sin h \beta} \right) \]

Where \( \beta = uL \) and \( u = \frac{P}{E l} \)

Figure 2: A 2D representation of the six degrees of freedom (coordinates) of a plane element in the xy plane