An Analysis of Two Approximate Techniques for Solution of Radioactive Materials Conversion Problem

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Abstract
This paper examines the conversion problem of five radioactive materials, the relationship and the effects of various radioactive parameters (decay constant and mean lifetime) on radioactive processes were also presented. A system of five coupled ordinary differential equations was obtained. The method of Variational Iteration Method and Homotopy Perturbation Method were applied to obtain the exact solutions to the system of coupled differential equations. The methods reduce the size of computations and do not require large computer resources and discretization of the variables. The effectiveness and accuracy of the methods were achieved as the results obtained rapidly converge after few numbers of iterations. The results obtained show that the mean life time of the radioactive materials decreases with time while the decay constant of the radioactive materials increases with time, also it was observed that, the higher the decay constant of a radioactive material, the faster the rate of conversion.

Keywords: variational iteration method, homotopy perturbation method, radioactive, decay, conversion

INTRODUCTION
According to Allisy, A. (1996) Radioactivity was discovered in 1896 by A. H Becquerel. Lilley J. (2001) describe radioactivity as a statistical process describing the spontaneous transformation of unstable atomic nuclei into a more stable configuration called daughter nuclei without the effect of physical and chemical condition. L’Annunziata, M. F. (2007) presented simplest radioactive decay where parent nuclide decays with a unique decay constant leading to a stable daughter nuclide. He proceeded, by presenting decay sequence where the decay of parent radionuclide forms a daughter product which is also itself radioactive. He formulated a set of three ordinary differential equations. Biazar, J. and Ebrahim, H. (2008) determined the solution of radioactive materials conversion problem. Three Radioactive materials were considered, they made use of Variational Iteration Method to determine the solution of three ordinary differential equation obtained from the conversion procedures. Radioactive Materials Conversion occurs when Radioactive Materials convert to another stable or unstable material through the process called Radioactive decay. Radionuclides are the sources of radioactivity and they emit nuclear radiations which have become a part of our daily life. The simplest radioactive decay is that of a radioactive parent nuclides decaying with a unique decay constant leading to a stable daughter nuclide. In this paper Variational Iteration Method and Homotopy Perturbation Method were used to solve the radioactive materials conversion problem of Five Materials. The Variational Iteration Method is one of the most versatile tools available for solving various kinds of linear and nonlinear equations. It was introduced by He in (1999, 2000(a), 2005, 2006(a)) while Homotopy Perturbation Method was first proposed by He in (2000(b)) the method does not depend on a small parameter in the equation.

He’s Variational Iteration Method
To illustrate the basic concept of Variational Method (VIM), we consider the following differential equation.
\[ Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \] (1)

Where L is a linear operator, R is the remaining linear operator, N a nonlinear operator and \( g(x,t) \) an inhomogeneous term. According to variational Iteration Method, we can construct functional as follows:
\[ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s) (L u_n(x,s) + R u_n(x,s) + N u_n(x,s) - g(x,s)) ds \]
\[ n \geq 0 \] (2)

Where \( \lambda(t,s) \) is a general Lagrange multiplier which can be identified optimally via the Variational theory. The function \( u_n \) is a restrictive variation i.e. \( \delta u_n = 0 \). Therefore we first determine the Lagrange multiplier that can be identified optimally via integration by parts. The successive
approximation \( u_{n+1}(x,t); n \geq 0 \) of the solution \( u(x,t) \) will be readily obtained upon using the Lagrange multiplier and by using any selective \( u_0(x,t) \). The zeroth approximation \( u_0(x,t) \) may be selected by any function that justifies at least two of the prescribed boundary condition. With \( \lambda \) determined, then several approximation \( u_{n+1}(x,t) \) follow immediately. Consequently, the exact solution may be obtained by using \( u(x,t) = \lim_{n \to \infty} u_n(x,t) \).

**Homotopy Perturbation Method**

To illustrate the basic ideas of this Method, the following nonlinear differential equation was considered:

\[
A(u) - f(r) = 0, \quad r \in \Omega, \quad (3)
\]

Subject to the boundary condition of:

\[
B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \quad (4)
\]

Where \( A \) is a general differential operator, \( B \) a boundary operator, \( f(r) \) a known analytical function and \( \Gamma \) is the boundary of the domain \( \Omega \). The operator \( A \) can be divided into two parts of \( L \) and \( N \), where \( L \) is the linear part, while \( N \) is the nonlinear one. (3) can therefore, be rewritten as follows:

\[
L(u) + N(u) - f(r) = 0, \quad r \in \Omega, \quad (5)
\]

The homotopy perturbation structure is shown as follows:

\[
H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad (6)
\]

Where:

\[
v(r,p) : \Omega \in [0,1) \to R \quad (7)
\]

In eqn. (2.6) \( p \in [0,1] \) is an embedding parameter and \( u_0 \) is the first approximation that satisfies the boundary condition. It can be assumed that the solution of eqn. (3) can be written as power series in \( p \) as follows:

\[
v = v_0 + pv_1 + p^2v_2 + \cdots \quad (8)
\]

And the best approximation for the solution is:

\[
u = \lim_{p \to 1} v = v_0 + pv_1 + p^2v_2 + \cdots \quad (9)
\]

The series (2.9) is convergent for most cases. However, the convergence rate depends on the nonlinear operator \( A(v) \). The following opinions are suggested by He (2000(b), 2006(b)).

The second derivative of \( N(v) \) with respect to \( v \) must be small because the parameter \( p \) may be relatively large, i.e. \( p \to 1 \).

The norm of \( L^{-1} \frac{\partial N}{\partial v} \) must be smaller than one so that the series converges.

**Governing Equations**

The radioactive Materials Conversion Procedure follows; material A converts to material B, and material B in turn converts to material C, and C in turn converts to material D, and D in turn converts to material E in similar procedure. The amount of materials A,B,C,D and E at time \( t \) are considered to be \( u, v, w, x \) and \( y \) respectively. Mathematical modeling of the problem consists of the following differential equations:

\[
\frac{du}{dt} = -k_1u \quad (10)
\]

\[
\frac{dv}{dt} = k_1u - k_2v \quad (11)
\]

\[
\frac{dw}{dt} = k_2v - k_3w \quad (12)
\]

\[
\frac{dx}{dt} = k_3w - k_4x \quad (13)
\]

\[
\frac{dy}{dt} = k_4x - k_5y \quad (14)
\]

Where \( k_1, k_2, k_3, k_4 \) and \( k_5 \) are respectively, rates of changes in the conversion procedure.

The Application of Variational Iteration Method for Radioactive Materials Conversion Problem

For solving (10) to (14) by Variational Iteration Method, correction functional can be written in the form:

\[
u_{n+1}(t) = u_n(t) + \int_0^t \lambda_1 \left[ \frac{du_n(s)}{ds} + k_1y_n(s) \right] ds \quad (15)
\]

\[
v_{n+1}(t) = v_n(t) + \int_0^t \lambda_2 \left[ \frac{dv_n(s)}{ds} - k_1u_n(s) + k_2v_n(s) \right] ds \quad (16)
\]

\[
w_{n+1}(t) = w_n(t) + \int_0^t \lambda_3 \left[ \frac{dw_n(s)}{ds} - k_2v_n(s) + k_3w_n(s) \right] ds \quad (17)
\]

\[
x_{n+1}(t) = x_n(t) + \int_0^t \lambda_4 \left[ \frac{dx_n(s)}{ds} - k_3w_n(s) + k_4x_n(s) \right] ds \quad (18)
\]

\[
y_{n+1}(t) = y_n(t) + \int_0^t \lambda_5 \left[ \frac{dy_n(s)}{ds} - k_4x_n(s) + k_5y_n(s) \right] ds \quad (19)
\]

Where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) and \( \lambda_5 \) are Lagrange multipliers. From (15) we obtain

\[
\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda_1 \left[ \frac{du_n(s)}{ds} + k_1y_n(s) \right] ds \quad (20)
\]

To make the function stationary with respect to \( u_n \) and that \( \delta u_n(0) = 0 \)

The stationary conditions are:

\[
\lambda_1 = 0 \quad (21)
\]

\[
\lambda_1 |_{s=t} = 0 \quad (22)
\]

\[
\lambda_1 = -1 \quad (23)
\]

Similarly we obtain \( \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1 \) and \( \lambda_5 = -1 \). By substituting \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) and \( \lambda_5 \) into
Using the identity

\[ \mathcal{J}(t) = \lim_{c \to \infty} \mathcal{J}(c) \]

(24) to (19), we obtain the following iterative formulas for system

\[ u_{n+1}(t) = u_n(t) - \int_0^t \left[ \frac{du_0(s)}{ds} k_1 u_n(s) \right] ds \]  

(25)

\[ v_{n+1}(t) = v_n(t) - \int_0^t \left[ \frac{dv_0(s)}{ds} - k_1 v_n(s) + k_2 v_n(s) \right] ds \]  

(26)

\[ w_{n+1}(t) = w_n(t) - \int_0^t \left[ \frac{dw_0(s)}{ds} - k_2 v_n(s) + k_3 w_n(s) \right] ds \]  

(27)

\[ x_{n+1}(t) = x_n(t) - \int_0^t \left[ \frac{dx_0(s)}{ds} - k_3 x_n(s) + k_4 x_n(s) \right] ds \]  

(28)

\[ y_{n+1}(t) = y_n(t) - \int_0^t \left[ \frac{dy_0(s)}{ds} - k_4 x_n(s) + k_5 y_n(s) \right] ds \]  

(29)

Starting with \( u_0(t) = u(0) = u_0 \) and \( \lambda_3 = -1 \) (30)

\[ u_1(t) = u_0(1-k_1 t) \]  

(31)

\[ u_2(t) = u_0(1-k_1 t + \frac{(k_1 t)^2}{2!}) \]  

(32)

\[ u_3(t) = u_0 \left[ 1 - k_1 t + \frac{(k_1 t)^2}{2!} - \frac{(k_1 t)^3}{3!} \right] \]  

(33)

\[ u_n(t) = u_0 \sum_{k=0}^n \left( \frac{(-k_1 t)^k}{k!} \right) \]  

(34)

Therefore, \( u(t) = \lim_{n \to \infty} u_n(t) \) (35)

Similarly starting with \( v_0(t) = v(0) = v_0 \), and \( \lambda_4 = -1 \)

\[ v_1(t) = \frac{k_1 u_0}{k_2 - k_1} \left[ (1 - k_1 t) - (1 - k_2 t) \right] + v_0(1 - k_2 t) \]  

(36)

\[ v_2(t) = u_0[k_1 t - (k_1^2 + k_1 k_2) \frac{t^2}{2!}] + v_0[1 - k_2 t + \frac{(k_2 t)^2}{2!}] \]  

(37)

Using the identity

\[ a^n + a^{n-1} b + a^{n-2} b^2 + \ldots + ab^{n-1} = \frac{a}{b-a} - \frac{a^n}{b^n} \]  

(38)

\[ v_n(t) = \frac{k_1 u_0}{k_2 - k_1} \left[ (1 - k_1 t) - \frac{(k_1 t)^2}{2!} \right] \]  

(39)

\[ + v_0 \sum_{k=0}^n \left( \frac{(-k_2 t)^k}{k!} \right) \]  

(40)

\[ + v_0 \sum_{k=0}^n \frac{(-k_2 t)^k}{k!} \]  

(41)

Therefore

\[ \psi(t) = \lim_{n \to \infty} \psi_n(t) \]  

(42)

\[ \psi(t) = \frac{k_1 u_0}{k_2 - k_1} \left[ e^{-k_1 t} - e^{-k_2 t} \right] + v_0 e^{-k_2 t} \]  

(43)

Similarly starting \( w_0(t) = w(0) = w_0 \) and \( \lambda_3 = -1 \)

\[ w_1(t) = k_2 v_0 + w_0 - k_3 w_0 \]  

(44)

\[ w_2(t) = \frac{1}{(k_3 - k_2)(k_3 - k_1) \sum_{k=0}^n \frac{(-k_1 t)^k}{k!}} + \frac{1}{(k_2 - k_1)(k_2 - k_3) \sum_{k=0}^n \frac{(-k_2 t)^k}{k!}} \]  

(45)

\[ w_3(t) = \frac{1}{(k_3 - k_1)(k_3 - k_2) \sum_{k=0}^n \frac{(-k_1 t)^k}{k!}} + \frac{1}{(k_2 - k_1)(k_2 - k_3) \sum_{k=0}^n \frac{(-k_2 t)^k}{k!}} \]  

(46)

\[ w_4(t) = k_1 k_2 u_0 \left[ \frac{1}{(k_3 - k_1)(k_3 - k_2) \sum_{k=0}^n \frac{(-k_1 t)^k}{k!}} \right] \]  

(47)

\[ + \frac{1}{(k_2 - k_1)(k_2 - k_3) \sum_{k=0}^n \frac{(-k_2 t)^k}{k!}} \]  

(48)

Again starting with \( x_0(t) = x(0) = x_0 \) and \( \lambda_4 = -1 \)

\[ x_1(t) = \frac{k_1 u_0}{k_2 - k_1} \left[ (1 - k_3 t) - (1 - k_4 t) \right] + x_0(1 - k_3 t) \]  

(49)

\[ x_2(t) = k_2 v_0 \left[ \frac{1}{(k_3 - k_2)(k_3 - k_4) \sum_{k=0}^n \frac{(-k_1 t)^k}{k!}} + \frac{1}{(k_2 - k_3)(k_2 - k_4) \sum_{k=0}^n \frac{(-k_2 t)^k}{k!}} \right] \]  

(50)

\[ + \frac{1}{(k_4 - k_3)(k_4 - k_5) \sum_{k=0}^n \frac{(-k_3 t)^k}{k!}} \]  

(51)

\[ + \frac{1}{(k_3 - k_2)(k_3 - k_4) \sum_{k=0}^n \frac{(-k_1 t)^k}{k!}} + k_3 w_0 \left[ \frac{1}{(k_2 - k_3)(k_2 - k_4) \sum_{k=0}^n \frac{(-k_2 t)^k}{k!}} \right] \]  

(52)
The Application of Homotopy Perturbation Method for Radioactive Materials Conversion Problem

Homotopy Perturbation Method is employ solve to (10) to (14)
\[(1 - p) \left( \frac{dw}{dt} - \frac{dw_0}{dt} \right) + \frac{dp}{dt} \left( \frac{dw}{dt} - k_2 w + k_3 w \right) = 0 \quad (85)\]

Considering
\[w = w + p w_1 + p^2 w_2 + p^3 w_3 \quad (86)\]
Substituting (86) into (85) and Equating the coefficient of equal power of \( p \), we have
\[p^0, w_0 = 0 \quad (87)\]
\[p^1; w_1 - k_2 v_0 + k_3 w_0 = 0 \quad (88)\]
\[p^2; w_2 - k_2 v_0 + k_3 w_1 = 0 \quad (89)\]
the solutions to (87) to (89) are obtained as
\[w_0 = w_0 \quad (90)\]
\[w_1 = (k_2 v_0 - k_3 w_0) t \quad (91)\]
\[W_2 = \frac{k_2^2 w_0 t^2}{2} - \frac{k_2 v_0 t^2}{2} - \frac{k_2 k_3 v_0 t^2}{2} + \frac{k_2^2 w_0 t^2}{2} \quad (92)\]

Substituting (90), (91) and (92) into (86), and As \( p \to 1 \) we have
\[w(t) = \frac{k_1 k_2 u_0 t^2}{2} + \left[ k_2 t - \left( k_2^2 + k_2 k_3 \right) \frac{t^2}{2} \right] + \left( 1 - k_3^2 \right) + \frac{k_2 v_0}{2} \quad (93)\]

Similarly, (93) using the identity (39) its transformation gives
\[w(t) = k_1 k_2 u_0 \quad (94)\]

Substituting (105), (106) into (105) and Equating the coefficient of equal power of \( p \), we have
\[p^0; y_0 = 0 \quad (107)\]
\[p^1; y_1 - k_4 x_0 + k_5 y_0 = 0 \quad (108)\]
\[p^2; y_2 - k_4 x_1 + k_5 y_1 = 0 \quad (109)\]
solutions to (107) to (109) are obtained as
\[y_0 = y_0 \quad (110)\]
\[y_1 = k_4 t x_0 - k_5 t y_0 \quad (111)\]
\[y_2 = \frac{k_3 k_4 w_0 t^2}{2} - \frac{k_3 v_0 t^2}{2} + \frac{k_3 k_4 x_0 t^2}{2} + \frac{k_3 v_0 t^2}{2} \quad (112)\]

Similarly, (113) using the identity (39) its transformation gives
\[y(t) = \frac{k_3 k_4 w_0 t^2}{2} + \left[ k_4 t - \left( k_4^2 + k_4 k_5 \right) \frac{t^2}{2} \right] + \left( 1 - k_5^2 \right) \quad (113)\]

RESULTS AND DISCUSSIONS
We present graphical summaries for various values of decay constants.
As shown in the figures (Figure 4.1 to Figure 4.5) the decay rate of material B depends on the decay rate of material A, C depends on B, D depends on C and E depends on D respectively and only material A is present at time $t = 0$. The mean lifetime of a radioactive material decreases with time and the decay constant of radioactive material increases with time. It was also observed that the lifetime of a radioactive materials decreases with time and the decay constant of radioactive materials increases with time.

**CONCLUSIONS**

The main goal of this work was the determination of solution for Radioactive Materials Conversion Problem. The methods of Variational Iteration Method and Homotopy Perturbation Method were used to find the solution to the set of differential equations. It is established that the methods are very efficient and rapidly converges as exact solutions are
obtained after few iterations. To avoid having unrealistic solution: it was assumed that the decay constant for the materials are different from each other, the initial amount of the materials is constant and the decay constant of the second materials is larger than the decay constant of the first materials. Finally the system of coupled equations were solved, relationships and effects of various radioactive parameters on radioactive processes were presented as shown in the results obtained.

REFERENCES


